

Critical Exponents of Superfluid Helium and Pseudo- ϵ Expansion

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Abstract

Pseudo- ϵ expansions (τ -series) for critical exponents of 3D XY model describing λ -transition in liquid helium are derived up to τ^6 terms. Numerical estimates extracted from the τ -series obtained using Padé-Borel resummation technique, scaling relations and seven-loop (τ^7) estimate for the Fisher exponent η are presented including those for exponents α and ν measured in experiments with record accuracy. For the exponent α the procedure argued to be most reliable gives $\alpha = -0.0117$. This number is very close to the most accurate experimental values differing appreciably from the results of numerous lattice and field-theoretical calculations. It signals that the pseudo- ϵ expansion is a powerful tool robust enough to evaluate critical exponents with very small absolute error. The arguments in favour of such a robustness are presented.

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I. INTRODUCTION

Nowadays there exists a great number of high-precision numerical estimates of critical exponents and other universal quantities for three-dimensional systems obtained within various theoretical approaches. High-temperature expansion technique, Monte Carlo simulations, field-theoretical renormalization-group analysis based upon many-loop calculations in three and $(4 - \epsilon)$ dimensions are among them (see, e. g. Refs.¹⁻³). In most cases, an agreement between theoretical estimates is so good and their (apparent) accuracy is so high that experimental results being, as a rule, less accurate start to lose their fundamental role in the physics of critical phenomena. Phrase "Experimentalists can only confuse us" said at the International Workshop in Bad Honnef⁴ sounds today even more actual than 20 years ago.

At the same time, there is an area within the phase transition science where experiment certainly passes ahead a theory. We mean the physics of superfluid transition in liquid helium-4. Traditionally⁵⁻⁷, experimental study of thermodynamic and kinetic properties of this quantum fluid is carried out on the very high technical level and covers the temperatures extremely close to the λ point. This, in particular, enabled to measure critical exponents of superfluid helium with unprecedented accuracy, including the specific heat exponent α which is known to be tiny. Record measurements⁸⁻¹⁰ performed in space in order to avoid the influence of gravity yielded $\alpha = -0.0127 \pm 0.0003$, the value of critical exponent that is accepted as the most accurate ever obtained experimentally.

In general, most of theoretical data agree or almost agree with the results of experimental determination of critical exponents for superfluid helium provided the uncertainty of computations is estimated in conservative enough way. Lattice estimates of α exhibit a tendency to group around -0.015 ¹¹⁻¹⁵ while their field-theoretical counterparts lie mainly between -0.004 and -0.013 ¹⁶⁻²⁴, i. e. oppositely regarding the experimental value mentioned. This discrepancy being small is nevertheless attracts attention (see, e. g. Refs.^{13,14,23}) and ways to resolve it are permanently looked for.

In such a situation it is resonable to evaluate the critical exponents for superfluid transition in helium-4 within an alternative approach which proved to be highly efficient numerically in the phase transition problem. We mean the method of pseudo- ϵ expansion invented by B. Nickel many years ago (see Ref. 19 in the paper of Le Guillou and Zinn-Justin¹⁸).

This method was applied to various systems^{25–31} including two dimensional and those with non-trivial symmetry of the order parameter and lead to rather good numerical results for all the models considered. High numerical power of pseudo- ϵ expansion technique stems from its key feature: it transforms strongly divergent renormalization group (RG) expansions into the series having smaller lower-order coefficients and much slower growing higher-order ones what makes them very convenient for getting numerical estimates. Moreover, as was recently shown³², the pseudo- ϵ expansion machinery works well even in the case of the Fisher exponent η when original RG expansion has irregular structure and is quite unsuitable for extracting numerical results.

Below, the pseudo- ϵ expansions (τ -series) for critical exponents of three-dimensional XY model will be calculated starting from the six-loop¹⁷ RG series. The τ -series for the exponents α and γ will be written down up to τ^6 terms. Numerical estimates for the critical exponents will be obtained using Padé-Borel resummation technique, scaling relations and the seven-loop (τ^7) pseudo- ϵ expansion estimate for Fisher exponent η . Comparing the numbers obtained with the results of the most advanced experiments and with the values extracted from lattice and field-theoretical calculations the numerical effectiveness of the pseudo- ϵ expansion approach will be evaluated. The general properties of this approach will be discussed and the roots of its high numerical power will be cleared up.

II. PSEUDO- ϵ EXPANSIONS FOR CRITICAL EXPONENTS α AND γ . NUMERICAL ESTIMATES

Critical thermodynamics of three-dimensional XY model is described by Euclidean field theory with the Hamiltonian:

$$H = \int d^3x \left[\frac{1}{2}(m_0^2\varphi_\alpha^2 + (\nabla\varphi_\alpha)^2) + \frac{\lambda}{24}(\varphi_\alpha^2)^2 \right], \quad (1)$$

where $\alpha = 1, 2$, bare mass squared m_0^2 is proportional to $T - T_c^{(0)}$, $T_c^{(0)}$ being the mean field transition temperature. Perturbative expansions for the β -function and critical exponents for the model (1) have been calculated within the massive theory^{17,33} with the propagator,

quartic vertex and φ^2 insertion normalized in a standart way:

$$\begin{aligned} G_R^{-1}(0, m, g_4) &= m^2, & \left. \frac{\partial G_R^{-1}(p, m, g_4)}{\partial p^2} \right|_{p^2=0} &= 1, \\ \Gamma_R(0, 0, 0, m, g) &= m^2 g_4, & \Gamma_R^{1,2}(0, 0, m, g_4) &= 1. \end{aligned} \quad (2)$$

We derive pseudo- ϵ expansions (τ -series) for critical exponents α and γ starting from corresponding six-loop RG series. To find these pseudo- ϵ expansions one has to substitute τ -series for Wilson fixed point coordinate g^* into perturbative RG series for critical exponents and reexpand them in τ . With τ -series for g^{*31} and RG expansions of exponents in hand the calculations are straightforward. Their results read:

$$\begin{aligned} \alpha &= \frac{1}{2} - \frac{3\tau}{10} - 0.1297777778\tau^2 - 0.039547352\tau^3 - 0.02432025\tau^4 \\ &\quad - 0.00324983\tau^5 - 0.0121092\tau^6. \end{aligned} \quad (3)$$

$$\begin{aligned} \gamma^{-1} &= 1 - \frac{\tau}{5} - 0.0405925926\tau^2 + 0.004326858\tau^3 - 0.00566467\tau^4 \\ &\quad + 0.00458218\tau^5 - 0.0067372\tau^6. \end{aligned} \quad (4)$$

We present here the pseudo- ϵ expansion for inverse γ instead of τ -series for the exponent γ itself because the former turns out to be more suitable for getting numerical estimates. We do not present τ -series for critical exponent ν since it can be easily deduced from (3) using well known scaling relation

$$\alpha = 2 - D\nu. \quad (5)$$

Despite of small and rapidly decreasing coefficients pseudo- ϵ expansions (3), (4) remain divergent. So, to extract numerical values of critical exponents from these series one has to apply some resummation procedure. We employ Padé-Borel resummation technique which is based on the Borel transformation

$$f(x) = \sum_{i=0}^{\infty} c_i x^i = \int_0^{\infty} e^{-t} F(xt) dt, \quad F(y) = \sum_{i=0}^{\infty} \frac{c_i}{i!} y^i. \quad (6)$$

and use of Padé approximants [L/M] for analytical continuation of the Borel transform $F(y)$. Application of this technique to τ -series for α leads, however, to the results which are far from to be satisfactory. This is seen from Table I representing Padé-Borel triangle for the series (3). More than a half of nontrivial estimates are absent in this table because of positive

axis ("dangerous") poles spoiling corresponding Padé approximants. Existing estimates are strongly scattered being practically useless for getting accurate value of the exponent α . Moreover, even more conservative procedure that uses simple Padé approximants and gives numerical results much less sensitive to the problem of poles results in numbers appreciably differing from each other even in the highest (τ^6) order available. Table II representing Padé triangle for the series (3) demonstrates this fact.

In such a situation it is natural to evaluate the exponent α in a different manner, using the scaling relation containing critical exponents γ and η . It is readily obtained combining (5) with

$$\nu = \frac{\gamma}{2 - \eta}. \quad (7)$$

This way to evaluate α looks attractive because of two reasons. First, Padé-Borel estimates of γ resulting from τ -series (4) converge to the asymptotic value very rapidly signaling that for this exponent the iteration procedure employed is rather efficient. This is clearly seen from Table III where the Padé-Borel triangle for the exponent γ is presented. Second, the numerical value of the Fisher exponent can be extracted from the recently found seven-loop τ -series³², i. e. it can be obtained with the highest accuracy accessible within the pseudo- ϵ expansion approach.

As is well known, diagonal and near-diagonal Padé approximants possess the best approximating properties³⁴. That is why the value $\gamma = 1.3156$ given by approximant $[3/3]$ (see Table III) is assumed to be the most reliable one resulting from the τ -series for γ^{-1} . Its counterpart originating from the τ -series for γ itself resummed within Padé-Borel technique using approximant $[3/3]$ is equal to 1.3162. So, the average over these two numbers will be accepted as a final pseudo- ϵ expansion estimate for the susceptibility critical exponent: $\gamma = 1.3159$. The pseudo- ϵ expansion estimate for η is extracted from seven-loop τ -series³²

$$\begin{aligned} \eta = & 0.0118518519\tau^2 + 0.0105390747\tau^3 + 0.005188190\tau^4 \\ & + 0.003229563\tau^5 + 0.00145159\tau^6 + 0.0016264\tau^7 \end{aligned} \quad (8)$$

by means of the same, Padé-Borel resummation procedure. Use of near-diagonal Padé approximant $[4/3]$ free of dangerous poles leads to $\eta = 0.0376$. This value agrees well with the results of alternative field-theoretical calculations $\eta = 0.0380(50)$ (ϵ -expansion), $\eta = 0.0370(50)$ (biased ϵ -expansion), and $\eta = 0.0354(25)$ (3D RG)²⁰ and is accepted as a pseudo- ϵ expansion estimate for the Fisher exponent.

It is worthy to evaluate the accuracy of numerical results thus found. To do this we adopt the following strategy. We assume that the difference between the numbers obtained from the same τ -series by means of various resummation procedures is a natural measure of numerical accuracy provided by the pseudo- ϵ expansion approach. In the case of critical exponents γ , use of simple Padé approximants to resum the series (4) gives the ultimate value 1.3154 deviating from above Padé-Borel estimate by 0.0005. This value is accepted to be a characteristic error of our estimate for γ . For the Fisher exponent Padé estimate originating from the seven-loop τ -series is equal to 0.0348³². Hence, the characteristic error in this case equals to 0.0028. For the exponent α evaluated via γ and η with a help of formulas (5) and (7) it leads to $\Delta\alpha = 0.0029$.

Using known scaling relations and estimating the accuracy of numerical results in the way just described we arrive to the following set of critical exponents for the λ -transition in liquid helium-4:

$$\alpha = -0.0117(29), \quad \nu = 0.6706(12), \quad \gamma = 1.3159(5), \quad \eta = 0.0376(28), \quad \beta = 0.3479(15). \quad (9)$$

Let us compare these values with experimental data and with the numbers extracted from field-theoretical and lattice calculations. Since critical exponent α is what is measured in experiments with highest accuracy¹⁰ along with the correlation length exponent⁷ related to α by the scaling relation (5) we concentrate here on the data for α . They are collected in Table IV. As is seen, our pseudo- ϵ expansion estimate is in a good agreement with the experimental data but deviates appreciably from the most of the results of RG analysis in three and $(4 - \epsilon)$ dimensions and from the lattice estimates. Hence, addressing the pseudo- ϵ expansion approach enables one to avoid discrepancy between theoretical predictions and the results of most accurate measurements.

III. PSEUDO- ϵ EXPANSION MACHINERY IS ROBUST

So, the pseudo- ϵ expansion approach results in iterations that converge to the high-precision values of critical exponents. It demonstrates a robustness of this approach that may be argued to be its general property. Indeed, let the pseudo- ϵ expansion for the Wilson

fixed point location g^* be:

$$g^* = \tau + A\tau^2 + B\tau^3 + C\tau^4 + D\tau^5 + \dots, \quad (10)$$

while field-theoretical RG series for some critical exponent ζ have a form:

$$\zeta = p_0 + p_1g + p_2g^2 + p_3g^3 + p_4g^4 + p_5g^5 + \dots \quad (11)$$

Then, to obtain τ -series for ζ , we have to substitute expansion (10) into (11). It yields:

$$\zeta = p_0 + p_1\tau + (Ap_1 + p_2)\tau^2 + (Bp_1 + 2Ap_2 + p_3)\tau^3 + [Cp_1 + (A + 2B)p_2 + 3Ap_3 + p_4]\tau^4 + \dots \quad (12)$$

As seen from (12) the coefficient of k -th term in the pseudo- ϵ expansion for ζ depends not only on the coefficients of the same order in series (10) and (11) but is determined by all the coefficients of k -th and lower orders starting from A and p_1 . It means that applying pseudo- ϵ expansion approach one uses the information contained in the known terms of original RG expansions to a greater extent than when conventional resummation procedures employed. This point is essential since the known coefficients of perturbative RG expansions are, in fact, the only input data we really use to evaluate the critical exponents and other universal quantities. All the rest information, e. g. the character of asymptotic behavior of coefficients under $k \rightarrow \infty$ is employed to choose the resummation procedure assumed to be optimal, to evaluate the (apparent) accuracy of numerical values obtained, etc., but not to fix the numbers themselves. Pseudo- ϵ expansion machinery realizing multiple use of the results of perturbative RG calculations is in this sense more robust than other resummation methods. High numerical efficiency of the pseudo- ϵ expansion approach demonstrated above may be referred to as manifestation of this robustness.

It is interesting that our estimate of α turns out to be close to that given by direct summation of corresponding τ -series (-0.0090). Although such a closeness may be thought of as occasional, it looks symptomatic and confirms the conclusion that pseudo- ϵ expansion is a robust procedure.

What would shed more light on the general properties and numerical power of the approach discussed is the knowledge of large-order behavior of pseudo- ϵ expansion coefficients. Today such information is absent. We believe, however, that it will be obtained in near future.

IV. CONCLUSION

To summarize, we have calculated pseudo- ϵ expansions for critical exponents of the three dimensional XY model up to τ^6 order. Numerical estimates have been found by means of Padé-Borel resummation of τ -series for the exponents γ combined with a use of scaling relations and numerical value of the Fisher exponent extracted from Padé-Borel resummed seven-loop τ -series. The values of critical exponents α and ν thus obtained turn out to be in a good agreement with the data of most precise measurements including those performed in space. It has been argued that pseudo- ϵ expansion approach represents resummation procedure that exploits the information contained in known coefficients of RG series to a greater extent than conventional resummation methods do. This may be the origin of its high numerical effectiveness that manifests itself, in particular, when critical exponents as tiny as the exponent α for superfluid helium are evaluated.

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TABLE I: Padé-Borel table for pseudo- ϵ expansion of critical exponents α . Many estimates are absent because corresponding Padé approximants are spoilt by dangerous (positive axis) poles.

$M \setminus L$	0	1	2	3	4	5	6
0	0.5000	0.2000	0.0702	0.0307	0.0064	0.0031	-0.0090
1	0.3456	—	—	—	—	—	
2	0.2927	0.0526	—	—	—		
3	0.2693	—	-0.0032	-0.0063			
4	0.2570	0.0534	—				
5	0.2498	—					
6	0.2452						

TABLE II: Padé triangle for the pseudo- ϵ expansion of critical exponents α .

$M \setminus L$	0	1	2	3	4	5	6
0	0.5000	0.2000	0.0702	0.0307	0.0064	0.0031	-0.0090
1	0.3125	-0.0287	0.0133	-0.0325	0.0026	0.0075	
2	0.2253	0.0161	-0.0025	-0.0035	-0.0082		
3	0.1769	-0.0212	-0.0035	-0.0023			
4	0.1451	0.0077	-0.0077				
5	0.1231	-0.0354					
6	0.1064						

TABLE III: Padé-Borel table for the critical exponent γ obtained from pseudo- ϵ expansion for γ^{-1} . Several estimates are absent because corresponding Padé approximants have poles on the positive real axis. The lowest line (RoC) demonstrates the rate of convergence of Padé-Borel estimates for γ to the asymptotic value. Here the estimate of k -th order is that given by diagonal approximant or by average over two near-diagonal ones when corresponding diagonal approximant does not exist.

$M \setminus L$	0	1	2	3	4	5	6
0	1	1.2500	1.3168	1.3094	1.3191	1.3112	1.3229
1	1.1736	—	1.3103	1.3133	1.3150	1.3157	
2	1.2363	1.3013	1.3159	1.3153	—		
3	1.2609	—	1.3152	1.3156			
4	1.2738	1.3047	—				
5	1.2908	—					
6	1.2853						
RoC	1	1.2118	1.2766	1.3058	1.3159	1.3152	1.3156

TABLE IV: The values of critical exponent α obtained in this work, found in experiments and extracted from resummed 3D RG series, ϵ -expansions and lattice calculations (LC).

This work	$-0.0117(29)$			
Experiments	$-0.0127(3)^{10}$	$-0.0115(18)^7$	$-0.0124(12)^6$	
3D RG	$-0.008(3)^{16}$	$-0.007(9)^{17}$	$-0.007(6)^{18}$	-0.010^{19}
	$-0.011(4)^{20}$	$-0.01294(60)^{21}$	$-0.01126(100)^{22}$	$-0.0100(20)^{23}$
ϵ -expansion	$-0.004(1)^{20}$	$-0.013^{20}(\text{biased})$	$-0.0091(39)^{24}$	
LC	$-0.0169(33)^{12}$	$-0.0146(8)^{13}$	$-0.0151(3)^{14}$	$-0.0151(9)^{15}$